COMBINATORICS

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1. Abstract

The study is a subjective review of 78 combinatorics tasks its authors have encountered during their professional careers. You will find here tasks that are easy and typical as well as non-trivial ones with brief and yet pleasantly ingenious solutions, which students interested in mathematics may find delight in. Some of them can be solved by presenting a combinatorial interpretation of one's own invention (in other words one's own story) or a graphic one. The number of tasks is not accidental either as there is also a story behind it.

2. Permutations

Permutations are arrangements of objects (with or without repetition), *order does matter*. The number of permutations of n objects, without repetition, is

$$P_n = P_n^n = n!$$

The counting problem is the same as putting n distinct balls into n distinct boxes, or to count bijections from a set of n distinct elements to a set of n distinct elements.
A permutation with repetition is an arrangement of objects where some objects are repeated a prescribed number of times. The number of permutations with repetitions of k_1 copies of 1, k_2 copies of 2, ..., k_r copies of r is

$$P_{k_1,\ldots,k_r} = \frac{(k_1 + \ldots + k_r)!}{\prod_{i=1}^{r} k_i!} = \frac{(k_1 + \ldots + k_r)!}{k_1! \cdot k_2! \cdot \ldots \cdot k_r!}$$

The counting problem is the same as putting $k_1 + \ldots + k_r$ distinct balls into r distinct boxes such that box number i receives k_i balls. In other words, we count onto functions from a set of $k_1 + \ldots + k_r$ distinct elements onto the set $\{1, 2, \ldots, r\}$, such that the preimage of the element i has size k_i.

Example 1. *In how many different ways can 10 people line up?*

Example 2. *In how many different ways can 10 people line up if Alice and Bob have to be next to each other?*

Example 3. *In how many different ways can 10 people line up if Alice and Bob can't be next to each other?*

Example 4. *In how many different ways can 10 people line up if Alice, Bob and Celine have to stand together?*

Example 5. *In how many different ways can 5 men and 5 women get on the bus?*

Example 6. *In how many different ways can 5 men and 5 women get on the bus if women enter first?*

Example 7. *In how many different ways can 5 men and 5 women get on the bus if men and women enter alternately?*

Example 8. *10 people are to sit at a round table; how many seating arrangements are there?*
First case: the chairs are distinguishable.
Second case: the chairs are not distinguishable and we care about the relative position of each person, it matters whether a certain person is on the left or right of another.
Third case: the chairs are not distinguishable and we care about who is next to whom, ignoring right and left.

Example 9. *5 men and 5 women are to be seated around a table, with men and women alternately. The chairs don't matter, only who is next to whom, but right and left are different. How many seating arrangements are possible?*

Example 10. *10 people are to be seated around a table; the chairs don't matter, only who is next to whom, but right and left are different. Two people, X and Y, have to be seated next to each other. How many seating arrangements are possible?*

3. Combinations

Combinations are selections of objects, with or without repetition, *order does not matter.* The number of k-element combinations of n objects, without repetition is

$$C_n^k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

The counting problem is the same as the number of ways of putting k identical balls into n distinct boxes such that each box receives at most one ball. It is also the number of k-element subsets of an n-element set.
The number of k-element combinations of n objects, with repetition is

$$\overline{C}_n^k = C_{n+k-1}^k = \binom{n+k-1}{k}$$

It is also the number of all ways to put k identical balls into n distinct boxes.
This is the only situation known to the authors where you can actually eat your cookie and have it too.

Let's consider set $A = \{1, 2, 3\}$ and list all its subsets with their corresponding binary sequences:

$$\begin{array}{cccccccc} \emptyset & \{1\} & \{2\} & \{3\} & \{1,2\} & \{1,3\} & \{2,3\} & \{1,2,3\} \\ (0,0,0) & (1,0,0) & (0,1,0) & (0,0,1) & (1,1,0) & (1,0,1) & (0,1,1) & (1,1,1). \end{array}$$

It's easy to see that

$$\sum_{k=0}^{k=n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} \cdots \binom{n}{n-1} + \binom{n}{n} = 2^n$$

$$(a+b)^n = \sum_{k=0}^{k=n} \binom{n}{k} a^k b^{n-k} = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n$$

Example 11. *Prove the following formulas:*

(a) $\binom{n}{k} = \binom{n}{n-k}$
(b) $k\binom{n}{k} = n\binom{n-1}{k-1}$
(c) $k \cdot (k-1)\binom{n}{k} = n \cdot (n-1)\binom{n-2}{k-2}$
(d) $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$
(e) $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$

Example 12. *Prove the following formula:*

$$\sum_{i=0}^{i=k} \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}$$

Example 13. *Prove the following formula:*

$$\sum_{k=1}^{k=n} k \binom{n}{k} = n \cdot 2^{n-1}$$

Example 14. *Prove the following formula:*

$$\sum_{k=2}^{k=n} k(k-1) \binom{n}{k} = n(n-1) \cdot 2^{n-2}$$

Example 15. *In how many ways can one choose a committee of 3 out of 10 people?*

Example 16. *A class consists of 25 students of whom 15 are girls and 10 are boys. How many committees of 5 can be formed if each consists of*

(a) *exactly 2 girls?*
(b) *at least 2 girls?*

Example 17. *There is a group of 6 men and 8 women. A committee consisting of 2 men and 3 women is formed. In how many ways can this be done if*

(a) *any man and any woman can be included?*
(b) *one particular woman must be on the committee?*
(c) *two particular men cannot be on the committee?*

Example 18. *In how many ways can 5 cards be chosen from a standard deck of 52 cards if*

(a) *exactly one of the cards is to be a Queen?*
(b) *at least two of the cards are diamonds?*
(c) *they make a full house in poker (full house is a hand that contains three cards of one rank and two cards of another rank)?*

Example 19. *In a chess tournament each of six players will play every other player exactly once. How many matches will be played during the tournament?*

Example 20. *Twelve children are to be divided into an A team and a B team of 6 each. The A team will play in one league and the B team in another. How many different divisions are possible?*

Example 21. *In order to play a game of volleyball, 12 children at a volleyball court divide themselves into two teams of 6 each. How many different divisions are possible?*

Example 22. *A group of 12 sport teams, among which there are teams A, B, C, is divided into 3 equal subgroups in such a way that each team A, B, C is in a different subgroup. How many ways of such a division are possible?*

Example 23. *Count the number of functions* $f : \{1, 2, 3\} \to \{1, 2, 3, 4, 5\}$

(a) *strictly increasing*
(b) *non-decreasing*
(c) *strictly decreasing*
(d) *non-increasing*

Example 24. *Count the number of functions* $f : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, n\}$

(a) *one-to-one*
(b) *strictly increasing*
(c) *non-decreasing*
(d) *strictly decreasing*
(e) *non-increasing*

Example 25. *Count the number of 5-digit numbers in which the digits are arranged*

(a) *ascending*
(b) *non-decreasing*
(c) *decreasing*
(d) *non-ascending*

Example 26. *Consider a chess board. In how many ways can you get from square A3 to square F7 if you can only move right or up?*

Example 27. *Consider a cartesian coordinate system. In how many ways can you get from point* $(0,0)$ *to point* (n,k) *if you can only move right or up?*

Example 28. *How many non-negative integer solutions does the equation:*
$x_1 + x_2 + \ldots + x_5 = 10$ *have?*

Example 29. *How many positive integer solutions does the equation:*
$x_1 + x_2 + \ldots + x_5 = 10$ *have?*

Example 30. *How many non-negative integer solutions does the equation:*
$x_1 + x_2 + \ldots + x_k = n$ *have?*

Example 31. *How many positive integer solutions does the equation:*
$x_1 + x_2 + \ldots + x_k = n$ *have?*

Example 32. *How many 15-digit numbers are those whose sum of digits is 7?*

4. Variations

Variations are arrangements of selections of objects, where the order of the selected objects matters. To count k-element variations of n objects, we first need to choose a k-element combination and then a permutation of the selected objects. Thus, the number of k-element variations of n elements with repetition not allowed is

$$V_n^k = \binom{n}{k} \cdot k!$$

It is also the number of ways of putting k distinct balls into n distinct boxes such that each box receives at most one element. It is also the number of one-to-one functions from a set of k distinct elements into a set of n distinct elements.
The number of k-element variations of n elements with repetition allowed, is

$$W_n^k = n^k$$

It is the number of all ways of putting k distinct balls into n distinct boxes. It is also the number of all functions from a set of k distinct elements into a set of n distinct elements.

Example 33. *How many different six-digit numbers can be constructed from the digits 1, 2, 3 ?*

Example 34. *How many different six-digit numbers can be constructed from the digits 0, 1, 2, 3 ?*

Example 35. *How many different four-digit numbers can be constructed from the digits 1, 2, 3, 4, 5, 6?*

Example 36. *How many different four-digit numbers can be constructed from the digits 0, 1, 2, 3, 4, 5, 6?*

Example 37. *How many different seven-digit numbers can be formed only from prime digits?*

Example 38. *How many different four-digit numbers are there?*

Example 39. *How many different four-digit PIN codes are there?*

Example 40. *How many different four-digit numbers are there without repetition?*

Example 41. *How many different four-digit PIN codes are there without repetition?*

Example 42. *How many different three-digit numbers can be formed with only odd digits?*

Example 43. *How many different three-digit numbers can be formed with only even digits?*

Example 44. *How many different three-digit numbers in which the digits do not repeat can be formed only from odd digits?*

Example 45. *How many different three-digit numbers in which the digits do not repeat can be formed only from even digits?*

Example 46. *How many three-digit odd numbers are there with distinct digits?*

Example 47. *How many three-digit even numbers are there with distinct digits?*

Example 48. *How many different three-digit numbers greater than 357 can be formed using the digits 2, 3, 4, 5, 6, 7 if the repetition of the digits is not allowed?*

Example 49. *How many four-digit numbers are divisible by 4?*

Example 50. *How many four-digit numbers are divisible by 25?*

Example 51. *How many four-digit numbers are divisible by 4 without repetition?*

Example 52. *How many four-digit numbers are divisible by 25 without repetition?*

Example 53. *How many flags with 3 differently colored horizontal stripes of the same width can be designed with 8 colors?*

5. Additional exercises

Dear Student, if you think that you can solve most of the above examples, try facing the following exercises.

Example 54. *In how many ways can the secretary put four letters addressed to different people into five different drawers?*

Example 55. *There are 4 people waiting at a railway platform. A train with 7 carriages arrives. In how many ways can the passengers get on the train if*

(a) *each passenger freely chooses the carriage to board?*
(b) *each passenger boards a different carriage?*
(c) *each passenger chooses the third, fourth or fifth carriage?*
(d) *passengers board exactly two carriages?*

Assume that we ignore the order of boarding.

Example 56. *How many ways can you arrange the letters of the word*

(a) *MATHS?*
(b) *PERMUTATIONS?*
(c) *COMBINATORICS?*
(d) *MATHEMATICS?*

Example 57. *How many different words can be formed of the letters of the word MATHEMATICS so that no two vowels are together?*

Example 58. *An absent-minded secretary prepares an envelope for each of five letters to different people. She then randomly puts the letters into the envelopes. In how many ways can she do it so that no recipient receives his letter?*

Example 59. *An absent-minded secretary prepares an envelope for each of n letters to different people. She then randomly puts the letters into the envelopes. In how many ways can she do it so that no recipient receives his letter?*

Example 60. *How many 5-digit numbers are there in which the digit 6 appears 3 times?*

Example 61. *Find number of five-digit numbers that have a digit that appears at least 3 times.*

Example 62. *From the set of digits 1,2,3,4,5,6,7 we create a 7-digit number. In how many ways can we do this so that the even numbers are not next to each other?*

Example 63. *How many seven-digit numbers are there in which the digit 7 appears twice, the units digit is 5, and the remaining digits are different and other than the abovementioned digits?*

Example 64. *How many 10-digit numbers are those whose sum of digits is 5?*

Example 65. *How many 10-digit numbers are those whose sum of digits is 5 and the digit 2 does not appear?*

Example 66. *How many 6-digit numbers are those whose product of digits is 63?*

Example 67. *From the set of digits 1,2,3,4,5,6,7,8,9 we create a 9-digit number. In how many ways can we do this so that the digits 1, 2, 3 are in ascending order?*

Example 68. *How many 6-digit numbers are divisible by 5?*

Example 69. *How many eight-digit numbers are there in which the digit 3 appears four times and the digit 5 appears twice?*

Example 70. *How many five-digit even numbers are there in which 0 appears at most twice?*

Example 71. *How many subsets of at least three elements are there of a six-element set?*

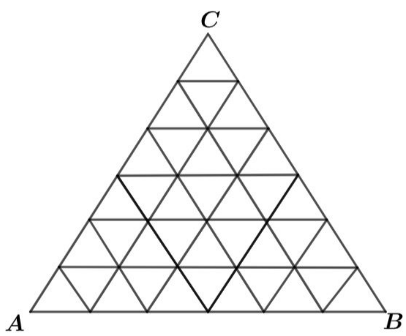
Example 72. *We have a 10-element set Z. How many pairs of sets (X, Y) are there such that $X \subset Y \subset Z$?*

Example 73. *There are eight numbered seats in a train carriage (in two rows opposite each other). Three women and two men entered the carriage. The women sat in one row and the men in the other. In how many ways can they take their seats so that each man sits opposite a woman?*

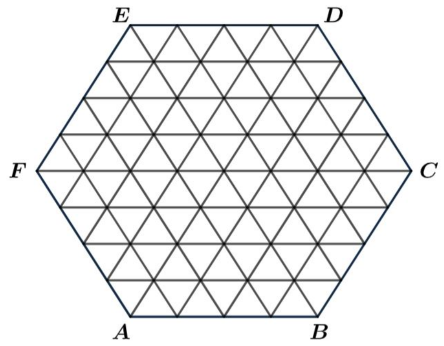
Example 74. *How many divisors does a number* $2^3 \cdot 3^4 \cdot 7^9$ *have?*

Example 75. *How many divisors does a number* 606375 *have?*

Example 76. *How many equilateral triangles whose vertices are lattice points can be formed in the equilateral triangle in the picture below? For the purposes of the task, lattice points are considered to be the intersection points of the lines in the figure below.*



Example 77. *How many regular hexagons whose vertices are lattice points can be formed in the regular hexagon in the picture below? For the purposes of the task, lattice points are considered to be the intersection points of the lines in the figure below.*



Example 78. *Sean wants to drive through the main campus of Trinity College in Dublin. However, there is a park inside where vehicle traffic is completely prohibited. The diagram of the streets on the campus and the location of the park are shown in the picture below. Sean is at point A on the campus and wants to take the shortest route to point B. Calculate how many shortest paths there are from point A to point B.*